THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 5 solutions

Compulsory Part

1. Let K and L be normal subgroups of G with $K \vee L = G$, and $K \cap L = \{e\}$. Show that $G/K \simeq L$ and $G/L \simeq K$.

   Answer. Let K and L be normal subgroups of G with $K \vee L = G$, and $K \cap L = \{e\}$. Then $G = K \vee L = KL = LK$. By the second isomorphism theorem, $G/K = KL/K \simeq L/L \cap K = L/\{e\} \simeq L$, and $G/L = KL/L \simeq K/K \cap L = K/\{e\} \simeq K$.

2. Suppose
$$\{e\} \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\varphi} K \to \{e\}$$
is an exact sequence of groups. Suppose also that there is a group homomorphism $\tau : G \to N$ such that $\tau \circ \iota = \mathrm{id}_N$. Prove that $G \simeq N \times K$.

   Answer. Define a map $\psi : G \to N \times K$ by $\psi(g) = (\tau(g), \varphi(g))$. We need to show that this map is an isomorphism.

   1. ψ is a homomorphism: Since τ and φ are group homomorphisms, so is ψ.

   2. ψ is injective: Suppose $\psi(g) = e_{H\times K}$. Then $\tau(g) = e$ and $\varphi(g) = e$. Since $\varphi(g) = e$, $g \in \ker(\phi) = \iota(N)$. Then $g = \iota(n)$ for some $n \in N$, and $n = \mathrm{id}_N(n) = \tau(\iota(n)) = \tau(g) = e$. Then $g = \iota(n) = e$. Therefore, $\ker(\psi) = \{e\}$ and ψ is injective.

   3. ψ is surjective: For every $(n, k) \in N \times K$, pick $g \in G$ such that $\varphi(g) = k$. Then $\psi(g) = (\tau(g), k)$. Let $n_1 = \tau(g)^{-1} n \in N$. Then $\varphi(g\iota(n_1)) = (\tau(g\iota(n_1)), \varphi(g\iota(n_1))) = (\tau(g)n_1, \varphi(g)) = (n, k)$. Therefore, ψ is surjective.

   Therefore, ψ is an isomorphism, so $G \simeq N \times K$.

3. Show that if
$$H_0 = \{e\} < H_1 < H_2 < \cdots < H_n = G$$
is a subnormal (normal) series for a group G, and if H_{i+1}/H_i is of finite order s_{i+1}, then G is of finite order $s_1 s_2 \cdots s_n$.

   Answer. Note that $\frac{|H_{i+1}|}{|H_i|} = |H_{i+1}/H_i| = s_{i+1}$. Therefore, $|G| = |G|/1 = \frac{|H_n|}{|H_0|} = \prod_{i=0}^{n-1} \frac{|H_{i+1}|}{|H_i|} = s_1 \ldots s_n$.

4. Show that an infinite abelian group can have no composition series.

   [*Hint:* Use the preceding exercise, together with the fact that an infinite abelian group always has a proper normal subgroup.]

Answer. First we show that an infinite abelian group cannot be simple: Let H be an infinite abelian group. Let $h \in H$ be an element different from e. If $\langle h \rangle$ is finite, then $\langle h \rangle < H$ is a nontrivial proper normal subgroup of H. If $\langle h \rangle$ is infinite, then $\langle h^2 \rangle \subsetneq \langle h \rangle \subseteq H$, and so $\langle h^2 \rangle$ is a nontrivial proper normal subgroup of H. In either case, H is not simple.

Now, let G be an infinite abelian group. Suppose that G has a composition series, that is, there is a subnormal series $H_0 = \{e\} < H_1 < H_2 < \cdots < H_n = G$ such that each H_{i+1}/H_i is simple. Then each H_{i+1}, being a subgroup of G, is abelian. Then each H_{i+1}/H_i is abelian, and so H_{i+1}/H_i is finite by the preceding paragraph. Then, by question 7, G is also of finite order. Contradiction arises. Therefore, G can have no composition series.

Remark. There are a lot of infinite simple groups. For example, $\mathrm{PSL}_n(k)$ $(n \geq 2)$ is simple whenever $n \geq 3$ or $|k| \geq 4$. In particular, $\mathrm{PSL}_n(k)$ is an infinite simple group for $|k| = \infty$.

5. Show that a finite direct product of solvable groups is solvable.

Answer. Let $G = G_1 \times ... \times G_n$, where each G_i is solvable. We prove by induction on n that G is solvable.

When $n = 1$, clearly, G is solvable. When $n \geq 2$, by induction hypothesis, $G' := G_1 \times ... \times G_{n-1}$ is solvable. Let $\{e\} = H_0 < H_1 < ... < H_a = G'$ be a subnormal series such that each H_{i+1}/H_i is abelian. Let $\{e\} = K_0 < K_1 < ... < K_b = G_n$ be a subnormal series such that each K_{i+1}/K_i is abelian. Then $H_0 \times K_0 < H_1 \times K_0 < ... < H_a \times K_0 < H_a \times K_1 < ... < H_a \times K_b$ is a subnormal series of $G = G' \times G_n$ such that each quotient is isomorphic to some H_{i+1}/H_i or some K_{i+1}/K_i, and is abelian. Then G is solvable.

Optional Part

1. Suppose N is a normal subgroup of a group G of prime index p. Show that, for any subgroup $H < G$, we either have

   - $H < N$, or
   - $G = HN$ and $[H : H \cap N] = p$.

   Answer. Suppose N is a normal subgroup of a group G of prime index p. Let $H < G$. Suppose H is not contained in N, then $HN/N < G/N$ is a nontrivial subgroup. Since $|G/N| = [G : N] = p$, $HN/N = G/N$. Therefore, $HN = G$, and by the second isomorphism theorem, $H/H \cap N \simeq HN/N$ has order p. Therefore, $[H : H \cap N] = p$.

2. Suppose N is a normal subgroup of a group G such that $N \cap [G, G] = \{e\}$. Show that $N \leq Z(G)$.

   [Hint: For $g \in G$, $n \in N$, $gng^{-1}n^{-1} \in N \cap [G, G] = \{e\}$.]

   Answer. Let $n \in N$ and $g \in G$, consider $gng^{-1}n^{-1}$, since N is normal, we have $gng^{-1} \in N$ and so $gng^{-1}n^{-1} \in N$. And also $gng^{-1}n^{-1}$ is a commutator so it is an element of $[G, G]$. By assumption, we have $gng^{-1} = e$, so $n \in Z(G)$.

3. Let $H_0 = \{e\} < H_1 < \cdots < H_n = G$ be a composition series for a group G. Let N be a normal subgroup of G, and suppose that N is a simple group. Show that the distinct groups among H_0, H_iN for $i = 0, \cdots, n$ also form a composition series for G.

   [*Hint:* Note that H_iN is a group. Show that $H_{i-1}N$ is normal in H_iN. Then we have

   $$(H_iN)/(H_{i-1}N) \simeq H_i/[H_i \cap (H_{i-1}N)],$$

   and the latter group is isomorphic to

   $$[H_i/H_{i-1}]/[(H_i \cap (H_{i-1}N))/H_{i-1}].$$

   But H_i/H_{i-1} is simple.]

   Answer. Let $H_0 = \{e\} < H_1 < \cdots < H_n = G$ be a composition series for a group G. Let N be a normal subgroup of G, and suppose that N is a simple group.

   For each i, $H_{i-1} \lhd H_i$, and $N \lhd H_iN$. Then for each $h \in H_i$, $hH_{i-1}h^{-1} = H_{i-1}$, and $hNh^{-1} = N$. Then $hH_{i-1}Nh^{-1} = hH_{i-1}h^{-1}hNh^{-1} = H_{i-1}N$. Then $H_i < N_G(H_{i-1}N)$, the normalizer of $H_{i-1}N$ in G. Clearly, $N < N_G(H_{i-1}N)$. Then $H_iN < N_G(H_{i-1}N)$, and so $H_{i-1}N \lhd H_iN$.

   Noting that $H_iN = H_i(H_{i-1}N)$, we have $H_{i-1}N \lhd H_i$ and $H_iN/H_{i-1}N \simeq H_i/(H_i \cap H_{i-1}N)$ by the second isomorphism theorem. Note that $H_{i-1} < H_{i-1}N$ are two normal subgroups of H_i, so we have $H_i/(H_i \cap H_{i-1}N) \simeq [H_i/H_{i-1}]/[(H_i \cap H_{i-1}N)/H_{i-1}]$ by the third isomorphism theorem. Therefore, $H_iN/H_{i-1}N$ is isomorphic to a quotient of H_i/H_{i-1}. Since each H_i/H_{i-1} is simple, so if $H_iN/H_{i-1}N$ is nontrivial, then it is isomorphic to H_i/H_{i-1}, and is simple.

   Now, the series $H_0 < H_0N < H_1N < ... < H_nN = G$ is a subnormal series, and each successive quotient is either trivial or simple. Therefore, selecting distinct groups among them will result in a composition series for G.

4. If H is a maximal proper subgroup of a finite solvable group G, prove that $[G : H]$ is a prime power.

   Answer. Note that the statement is true when G is a finite abelian group, this follows from classification of finitely generated abelian group, whereby one can write down explicit maximal proper subgroups, all of which have prime index.

   A subgroup N is called minimal normal if it is nontrivial and normal in G and the only proper subgroup $M \leq N$ which is normal in G is the trivial subgroup. Note that $N' = [N, N]$ is a normal subgroup of G, since $g[n_1, n_2]g^{-1} = [gn_1g^{-1}, gn_2g^{-1}]$ for $g \in G$, $n_1, n_2 \in N$. By solvability of G and minimality of N, we know N' is trivial, i.e. N must be abelian. Let p be a prime that divides $|N|$, consider the subgroup $M = \{n \in N | \, n^p = e\}$ of N, again this is a normal subgroup of G (check this). By Cauchy's theorem (see lecture 5), $M \neq \{e\}$, so by minimality of N, we have $M = N$, i.e. every $n \in N$ satisfies $n^p = e$. By classification of finitely generated abelian groups, this implies that $N \cong (\mathbb{Z}/p\mathbb{Z})^k$ for some $k \geq 1$. Such groups are called elementary abelian groups.

   Now we prove the statement using induction. Consider $H \leq HN \leq G$, by normality of N, we know HN is a subgroup. The case for which G is abelian is known as explained before, so from now we assume that G is nonabelian but solvable. Then N is proper since for example $[G, G]$ is normal. By maximality of H, we know $HN = H$ or $HN = G$. In the first case $HN = H$, consider $HN/N = H/N$, it is solvable since it is a subgroup of G/N, which is also solvable. Also, H/N is a maximal subgroup of G/N, which has strictly smaller order than G since N is proper. By induction hypothesis, $[G/N : H/N]$ is a prime power, but this index is equal to $[G : H]$, so we are done. In the second case, $HN = G$, so $[G : H] = [HN : H] = [N : H \cap N]$ by second isomorphism theorem. By the above result in the previous paragraph, we know $[N : H \cap N]$ is a prime power.